PAVOL JOZEF ŠAFÁRIK UNIVERSITY IN KOŠICE
Faculty of Science

UNCERTAINTY MODELING 2024

Book of Abstracts

May 24 – 25, 2024

Ondrej Hutník (ed.)

Košice 2024

Edited by:

prof. RNDr. Ondrej Hutník, PhD.

Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University in Košice

Organized by:

Pavol Jozef Šafárik University in Košice

org.com (civil organization)

All papers have been reviewed.

Available at: `https://unibook.upjs.sk`

ISBN 978-80-574-0318-0 (e-publication)

Preface

Uncertainty Modeling 2024 (UM 2024) is organized by Pavol Jozef Šafárik University in Košice in cooperation with the civil organization org.com. It is a continuation of the series of colloquia held in Rzeszów under the name International Symposium on Fuzzy Sets (ISFS) and in Bratislava under the name Uncertainty Modeling.

The main goal of the workshop is to bring together researchers, doctoral candidates, and students from various countries (most frequently Slovakia, Poland, and the Czech Republic) working in the field of modeling uncertainty. This encompasses a wide range of topics such as aggregation functions, fuzzy logic, and fuzzy sets. By fostering collaboration and knowledge exchange among participants, the workshop aims to advance the understanding and application of these important mathematical and computational concepts in addressing uncertainty in various scientific and practical domains.

The history of the event dates back to 1990, when a regular scientific seminar was initiated by Prof. Beloslav Riečan and Prof. Radko Mesiar. Since 1995, it has been hosted by the Department of Mathematics and Descriptive Geometry of the Faculty of Civil Engineering at the Slovak University of Technology in Bratislava. Since 2014, the event has become part of international workshops organized annually by institutions such as STU Bratislava and the University of Rzeszów. The idea to organize the workshop in Košice was born in Bratislava in 2022 after the difficult pandemic years. A group of young researchers took the initiative to organize the international workshop at Pavol Jozef Šafárik University in Košice. The honorary patronage of the workshop is held by the Dean of the Faculty of Science, Professor Roman Soták. The organizers would also like to thank the faculty for its support.

We hope the recent workshop will provide an excellent international forum for sharing knowledge and recent results. Our gratitude goes to all the authors, as well as to the members of the Programme Committee, whose work has contributed to the high quality of the contributions included in these Proceedings. We especially thank the invited speakers, Ľubomíra Horanská and Michał Baczyński, for their interesting lectures, which have greatly enriched the workshop and inspired further research and discussions.

Košice, May 20, 2024

Ondrej Hutník

Uncertainty Modeling 2024 editor

Contents

On the multiplicative Sincov's equation for fuzzy connectives

BACZYŃSKI Michał

University of Silesia in Katowice, Faculty of Science and Technology, ul. Bankowa 14, 40-007 Katowice, Poland, email: michal.baczynski@us.edu.pl

Fuzzy implication functions are one of the main mathematical operations in fuzzy logic. They generalize the classical two-valued implication to fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. This family of functions plays a significant role in the development of fuzzy systems, see [2]. Recently, the so-called family of power based implications was introduced in [3] as a new family in which most of its members satisfy the invariance with respect to powers of a continuous t-norm [3, 4], an important additional property in approximate reasoning. The class of power based implications was characterized through, among others, the multiplicative Sincov's equation $I(x, y) \cdot I(y, z) = I(x, z)$ in a concrete sub-domain, see [5]. In our presentation we plan to present the most recent results concerning the multiplicative Sincov's equation, as well as some its generalizations, cf. [1].

References

[1] Baczyński, M., Fechner, W., Massanet, S.: On a generalization of multiplicative Sincov's equation for fuzzy implication functions. *Fuzzy Sets and Systems* **451** (2022), 196–205.

[2] Baczyński, M., Jayaram, B.: *Fuzzy Implications*, Vol. **231** of *Studies in Fuzziness and Soft Computing.* Springer, Berlin Heidelberg, 2008.

[3] Massanet, S., Recasens, J., Torrens, J.: Fuzzy implication functions based on powers of continuous t-norms. *Internat. J. Approx. Reason.* **83** (2017), 265–279.

[4] Massanet, S., Recasens, J., Torrens, J.: Corrigendum to fuzzy implication functions based on powers of continuous t-norms [Int. J. Approx. Reason. 83 (2017) 265–279]. *Internat. J. Approx. Reason.* **104** (2019), 144–147.

[5] Massanet, S., Recasens, J., Torrens, J.: Some characterizations of T-power based implications. *Fuzzy Sets and Systems* **359** (2019), 42–62.

Möbius transform: History, generalizations and applications in aggregation theory

HORANSKÁ Ľubomíra

Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology in Bratislava, Radlinského 9, 812 37 Bratislava, Slovak Republic, email: lubomira.horanska@stuba.sk

The classical inclusion-exclusion principle as a particular case of inversion of a finite series was studied by several mathematicians in 19th century (e.g. Poincaré) and in early 20th century (Dedekind, Liouville, Fréchet etc.). It is closely related to the notion of Möbius transform which appeared for the first time in the field of number theory in 1832 [1] when A. F. Möbius investigated the relation between a pair of arithmetic functions the first of which can be additively represented by the values of the second over all divisors of the argument. The inverse relation was expressed with the aid of a function later named Möbius function which captures the structure of the divisor poset. The Möbius inversion formula in more general settings for real valued functions defined on finite partially ordered sets was independently discovered by Weisner and Hall in the 1930s [2, 3], albeit with some redundant assumptions and used only for a group theory problems.

In 1964 Rota [4] pointed out that the problem of inverting summation is connected to many problems (not only) in combinatorial mathematics. While Möbius function capturing the structure of the underlying poset is an analog of the difference operator, the Möbius inversion formula can be seen as an analog of fundamental theorem of the calculus. Rota deeply studied properties of the Möbius functions in settings of locally finite posets using the so-called incidence algebra and applied his results, for instance, in graph coloring and flows in network problems.

In aggregation theory, especially for purposes of decision making and artificial intelligence, the concept of fuzzy measures is of particular interest. Fuzzy measures are functions defined on a power set partially ordered by inclusion (Boolean lattice) and they reflect the interaction between decision criteria. Their identification can be a challenging task, therefore representations of fuzzy measures by means of Möbius transform have proven to be very useful [5]. This has been the driving force for many generalizations of the Möbius transform [6] and, on the other hand, for its use to reduce the complexity of the identification task [7, 8]. Finally, it should also be mentioned that for the same reason, the constructions of Möbius transform-based constructions of aggregation functions are also particularly interesting [9, 10]. Recently, in the work [11], a concept of Möbius product covering many such constructions and providing a large space for further research appeared. We focus on two particular cases of Möbius product-based construction, the first one considering only 2-additive fuzzy measures which are very common in real-life scenarios and the second one dealing with extended aggregation functions rather then aggregation functions with fixed arity, which impose significant constraints in Lovasz-like constructions.

Acknowledgement.
Support of the project VEGA 1/0239/24 is kindly acknowledged.

References

[1] Möbius, A. F.: Über eine besondere Art von Umkehrung der Reihen. Journal für die reine und angewandte Mathematik **9** (1832), 105–123. ISSN 0075-4102. `doi.org/10.1515/crll.1832.9.105`

[2] Hall, P.: A contribution to the theory of groups of prime power order. Proc. London Math. Soc. **36** (1934), 24–80. `doi.org/10.1112/plms/s2-36.1.29`

[3] Weisner, L.: Abstract theory of inversion of finite series. Trans. Amer. Math. Soc. **38** (1935), 474–484. `doi.org/10.1090/S0002-9947-1935-1501822-0`

[4] Rota, G.-C.: On the foundations of combinatorial theory I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie **2** (1964), 340–368.

[5] Chateauneuf, A., Jaffray, J. Y.: Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. Math. Social Sci. **17** (1989), pp. 263–283. ISBN 978-3-540-25381-5. `doi.org/10.1016/0165-4896(89)90056-5`

[6] Mesiar, R.: Fuzzy measures and generalized Möbius transform. Internat. J. Gen. Sys. **31** (2002), 587–599. ISSN 0308-10791563-5104. `doi.org/10.1080/0308107021000034029`

[7] Grabisch, M.: k-order additive discrete fuzzy measures and their representation. Fuzzy Sets and Systems **92** (1997), 167-189. ISSN 0165-0114. `doi.org/10.1016/S0165-0114(97)00168-1`

[8] Mesiar, R.: Generalization of k-order additive discrete fuzzy measures. Fuzzy Sets and Systems **102** (1999), 423–428. ISSN 0165-0114. `doi.org/10.1016/S0165-0114(98)00216-4`

[9] Lovász, L.: Submodular function and convexity, in: *Mathematical Programming: The State of the Art*, Springer, Berlin, (1983), pp. 235–257. ISBN 978-3-642-68876-8. `doi.org/10.1007/978-3-642-68874-4-10`

[10] Kolesárová, A., Stupňanová, A., Beganová, J.: Aggregation-based extensions of fuzzy measures. Fuzzy Sets and Systems **194** (2012), 1–14. ISSN 0165-0114. `doi.org/10.1016/j.fss.2011.11.003`

[11] Mesiar, R., Kolesárová, A., Borkotokey, S., Jin, L.: Möbius product-based constructions of aggregation functions. Fuzzy Sets and Systems **448** (2022), pp. 17–34. ISSN 0165-0114. `doi.org/10.1016/j.fss.2022.01.002`

Extension of fuzzy attribute implications for heterogeneous formal context and its applications

ANTONI Ľubomír · ELIAŠ Peter · GUNIŠ Ján · KOTLÁROVÁ Dominika · KRAJČI Stanislav · KRÍDLO Ondrej · ŠNAJDER Ľubomír
Institute of Computer Science, Faculty of Science, Pavol Jozef Šafárik University in Košice, Jesenná 5, 040 01 Košice, Slovakia, email: {lubomir.antoni, jan.gunis, stanislav.krajci, ondrej.kridlo, lubomir.snajder}@upjs.sk, dominika.kotlarova@student.upjs.sk · Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, Košice, 04001, Slovakia, email: elias@saske.sk

Methods of Formal concept analysis and their fuzzy extensions, based on results from applied algebra, fuzzy logic, aggregation operators, or conceptual structures, are effective tools in computer science, data analysis, or applied mathematics [1, 2].

Current research in Formal concept analysis focuses, for example, on the attribute implications [3], visualization of outputs [4], matrix factorization [5], or fuzzy relational Galois connections [6].

In computer science, or applied mathematics, the implications

$$U \Rightarrow V$$

over a set A, $U \subseteq A$, and $V \subseteq A$ have been thoroughly explored [7].

Hájek [8] published the first prospects about attribute implications in fuzzy setting. Moreover, the first frameworks on fuzzy attribute implications in Formal concept analysis were given by Bělohlávek and Vychodil [9].

Our research in heterogeneous extensions [10] continues to investigate the attribute implications and their validity in our previously proposed heterogeneous formal context. In particular, we present the definition of heterogeneous (fuzzy) attribute implications.

Consider a set A of attributes. Let F be a set of all heterogeneous fuzzy subsets of A. We will call a pair (f, h) by a heterogeneous attribute implication over A if $f, h \in F$.

Moreover, we explored the validity of the heterogeneous attribute implications in a formal context and derived their properties regarding the connections with the intents of heterogeneous formal concepts. We present the applications of selected types of attribute implications in various domains. The Formal concept analysis not only sets a theoretical foundation for use across multiple fields but also enables novel applications in various domains.

Acknowledgement.
This work was supported by the Slovak Research and Development Agency under contract No. APVV-21-0468 (Ľ. Antoni, S. Krajči, O. Krídlo).

References

[1] Ganter, B., Wille, R.: *Formal Concept Analysis, Mathematical Foundation.* Springer Verlag, 1999. ISBN: 978-3-642-59830-2. doi:10.1007/978-3-642-59830-2

[2] Missaoui, R., Kwuida, L., Abdessalem, T.: *Complex Data Analytics with Formal Concept Analysis*. Springer, Cham, 2022. ISBN: 978-3-030-93278-7. doi:10.1007/978-3-030-93278-7

[3] Cornejo, M.E., Medina, J., Ocaña, F.J.: Attribute implications in multi-adjoint concept lattices with hedges. Fuzzy Sets and Systems **479**, (2024), 108854. ISSN 1872-6801. doi:10.1016/j.fss.2023.108854

[4] Priss, U.: Representing concept lattices with Euler diagrams. In *International Conference on Formal Concept Analysis*, (2023), pp. 183-197. ISBN: 978-3-031-35948-4. doi:10.1007/978-3-031-35949-1_13

[5] Bělohlávek, R., Trnečková, M.: Factorization of Matrices With Grades With Overcovering. IEEE Transactions on Fuzzy Systems **32**(4), (2024), pp. 1641-1652. ISSN 1063-6706. doi:10.2139/ssrn.4446715

[6] Cabrera, I.P., Cordero, P., Muñoz-Velasco, E., Ojeda-Aciego, M., De Baets, B.: Fuzzy relational Galois connections between fuzzy transitive digraphs. Fuzzy Sets and Systems **463**, (2023), 108456. ISSN 1872-6801. doi:10.1016/j.fss.2022.12.012

[7] Ojeda-Hernández, M., Pérez-Gámez, F., López-Rodríguez, D., Madrid, N., Mora, Á.: Minimal Generators from Positive and Negative Attributes: Analysing the Knowledge Space of a Mathematics Course. International Journal of Computational Intelligence Systems **15**(1), (2022), 58. ISSN 1875-6883. doi:10.1007/s44196-022-00123-3

[8] Hájek, P.: *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998. ISBN: 978-0-7923-5238-9 . doi:10.1007/978-94-011-5300-3

[9] Bělohlávek, R., Vychodil, V.: Attribute implications in a fuzzy setting. In *Proceedings of ICFCA 2006*, (2006), pp. 45-60. ISBN: 978-3-540-32203-0. doi:10.1007/11671404_3

[10] Antoni, Ľ., Krajči, S., Krídlo, O., Macek, B., Pisková, L.: On heterogeneous formal contexts. Fuzzy Sets and Systems **234**, (2014), pp. 22-33. ISSN 1872-6801. doi:10.1016/j.fss.2013.04.008

Towards time-dependent scientometric indices: Sugeno chain-based integrals

BORZOVÁ Jana · KLEINOVÁ Miriam
Institute of Mathematics, Pavol Jozef Šafárik University in Košice, Jesenná 5, 040 01 Košice, Slovakia,
email: jana.borzova@upjs.sk · miriam.kleinova@student.upjs.sk

In a recent paper by Stupňanová [7], several new time-dependent scientometric indices were introduced and exemplified. These indices build upon the original considerations of Hirsch [4], Kosmulski [5], and Egghe [3], extending them to describe the time-dependent performance of scholars. This novel approach not only considers the number of citations received by a scholar's articles but also takes into account the time elapsed since publication of each article. In Stupňanová's work, three generalizations of the Hirsch $\mathfrak{h}$-index (namely $\mathfrak{h}_T$, $\mathfrak{h}^T$, and $\mathfrak{h}_T^T$), three generalizations of the Kosmulski $\mathfrak{k}$-index (namely $\mathfrak{k}_T$, $\mathfrak{k}^T$, and $\mathfrak{k}_T^T$), and one generalization of the Egghe $\mathfrak{g}$-index (namely $\mathfrak{g}_T$) can be found. Towards the end of the paper, the author states an open problem regarding the introduction of the remaining two time-dependent indices, $\mathfrak{g}^T$ and $\mathfrak{g}_T^T$.

Motivated by this, we introduce a new operator acting on the set of all nonnegative real-valued vectors $\mathbf{x} \in \mathbf{F}_n$ with respect to a monotone grounded set function $\mu \in \mathbf{M}_n$. This operator is of the form

$$\mathbf{CSu}^{\oplus,\otimes}_{\mathscr{A},\mathcal{C}}(\mu,\mathbf{x}) := \bigoplus_{i\in[m]} \Big(\mu(C_i) \otimes \mathsf{A}(\mathbf{x}|C_i)\Big). \tag{1}$$

Here, $\oplus$ and $\otimes$ denote m-ary and binary operations, respectively, $\mathscr{A}$ represents a family of conditional aggregation operators, and $\mathcal{C} \in \mathscr{C}_m([n])$ is a chain on the discrete basic set $[n]$. This approach allows us to provide an integral representation of the $\mathfrak{g}$-index, as well as to define the time-dependent indices $\mathfrak{g}^T$ and $\mathfrak{g}_T^T$ in the spirit of [7].

Despite this, we present several interesting properties of the operator (1) and its relationship with existing objects, such as the twofold integral [6] and the Sugeno-like FG-functional [1]. Additionally, we discuss the equivalent representation of (1) via the recently introduced generalized level measure, see [2].

Acknowledgement.
The support of the grant APVV-21-0468 and VEGA 1/0657/22 is kindly acknowledged.

References

[1] Bardozzo, F., De La Osa, B., Horanská, Ľ., Fumanal-Idocin, J., delli Priscoli, M., Troiano, L., Tagliaferri, R., Fernandez, J., Bustince, H.: Sugeno integral generalization applied to improve adaptive image binarization. *Inform. Fusion* **68** (2021), 37–45. https://doi.org/10.1016/j.inffus.2020.10.020

[2] Boczek, M., Hutník, O., Kaluszka, M., Kleinová, M.: Generalized level measure based on a family of conditional aggregation operators. *Fuzzy Sets and Systems* **457** (2023), 180-196. https://doi.org/10.1016/j.fss.2022.07.011

[3] Egghe, L.: Theory and practise of the g-index. *Scientometrics* **69** (2006), 131–152. https://doi.org/10.1007/s11192-006-0144-7

[4] Hirsch, J.E.: An index to quantify an individual's scientfic research output. *Proc. Nat. Acad. Sci.* **102**(4) (2005), 16569–16572. https://doi.org/10.1073/pnas.0507655102

[5] Kosmulski, M.: MAXPROD-a new index for assessment of the scientific output of an individual, and a comparison with the h-index. *Cybermetrics: International Journal of Scientometrics, Informetrics and Bibliometrics* **11**(1) (2007), 1-5.

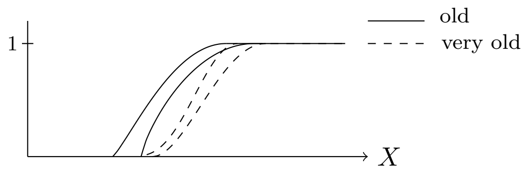
[6] Narukawa, Y., Torra, V.: Twofold integral and multi-step Choquet integral. *Kybernetika* **40** (2004), 39–50.

[7] Stupňanová, A.: Author-level metrics dependent on time. *Fuzzy Sets and Systems* **477** (2024), 108795. https://doi.org/10.1016/j.fss.2023.108795

Interval-valued relational modifiers

BRUTENIČOVÁ Michaela
Matej Bel University, Banská Bystrica, Slovakia, email: michaela.brutenicova@umb.sk

One of applications of fuzzy sets is in linguistic theory. Any fuzzy set could represent a linguistic variable, for example "old". Another value "very old" is a kind of a modified variable.



In our case a modifier m is a transformation of a set to another set. The research is focused on classical fuzzy sets ($A : X \to [0,1]$) and lattice-valued fuzzy sets ($A : X \to L$), specifically on interval-valued fuzzy sets ($A : X \to L([0,1])$), shortly IVFSs.

An introductory overview of fuzzy modifiers can be found in [5]. Our focus lies on a specific category of fuzzy modifiers, originally introduced in [3] for classical fuzzy sets.

For X a universe, R a fuzzy relation on X, $\mathcal{C}$ a conjunctor and $\mathcal{I}$ an implicator, we define the fuzzy modifiers $R_{\mathcal{C}}$ and $R_{\mathcal{I}}$ on X with $A \in \mathcal{F}(X)$

$$\begin{aligned} R_{\mathcal{C}}(A) : X &\longrightarrow [0,1] \\ y &\longmapsto \sup_{x \in X} \mathcal{C}(R(x,y), A(x)), \quad \forall y \in X \end{aligned}$$

$$\begin{aligned} R_{\mathcal{I}}(A) : X &\longrightarrow [0,1] \\ y &\longmapsto \inf_{x \in X} \mathcal{I}(R(x,y), A(x)), \quad \forall y \in X \end{aligned}$$

We study a generalization of these modifiers for IVFSs and the preservation of convexity. When dealing with IVFSs, we investigate convexity under various orders such as lattice order, lexicographical order of type 1 and 2, Xu and Yager order, interval dominance, and others. Further information on these orders can be found, for instance, in [4]. Each order $\preceq$ induces its corresponding $\preceq$-convexity.

This investigation is motivated by our previous research on another class of modifiers ([2]).

References

[1] Bustince H., Burillo P.: Mathematical analysis of interval-valued fuzzy relations: Application to approximate reasoning. Fuzzy Sets and Systems **113**, (2000), pp. 205-219. ISSN 0165-0114. doi.org/10.1016/S0165-0114(98)00020-7

[2] Bruteničová M., Janiš V.: Postmodifiers preserving monotony. In *International Symposium on Fuzzy Sets: Book of Abstracts*, (2023), p. 26. ISBN: 978-83-8277-077-3.

[3] De Cock M., Kerre E. E.: Fuzzy modifiers based on fuzzy relations. Information Sciences **160**,(2004),pp. 173-199. ISSN 0020-0255. doi.org/10.1016/j.ins.2003.09.002

[4] Huidobro P., Alonso P., Janiš V., Montes S.: Orders Preserving Convexity Under Intersection for Interval-Valued Fuzzy Sets. In: *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, (2020), pp. 493-505. doi: 10.1007/978-3-030-50153-2_37

[5] Kerre, E.E., De Cock, M.: Linguistic Modifiers: An Overview. In: Chen, G., Ying, M., Cai, KY. (eds) *Fuzzy Logic and Soft Computing. The International Series on Asian Studies in Computer and Information Science, vol 6.*, (1999) Springer, Boston, MA. ISBN: 978-1-4615-5261-1. doi.org/10.1007/978-1-4615-5261-1_5

New types of threshold generated ordinal sums of fuzzy implications

DRYGAŚ Paweł · KRÓL Anna
College of Natural Sciences, University of Rzeszów, Pigonia 1, 35-310 Rzeszów, Poland, email: {padrygas, akrol}@ur.edu.pl

A fuzzy implication as one of the connectives of fuzzy logic is used in many areas, e.g. in fuzzy approximate reasoning, in various inference schemes. Many families of the fuzzy implications were proposed and studied ([1]).

We present a construction that allows us to generate a fuzzy implication from a certain family of fuzzy implications. The construction is based on the ones proposed by Massanet and Torrens ([2, 3]) and the one introduced by Lima, Bedregal and Mezzomo ([4]). We examine the properties of the new operation depending on the properties of its generators. Moreover, we present some comparisons to existing methods of constructions.

References

[1] Baczyński, M., Jayaram, B.: *Fuzzy implications*, Springer, 2008. ISBN: 978-3-540-69080-1. doi:10.1007/978-3-540-69082-5

[2] Massanet, S., Torrens, J.: Threshold generation method of construction of a new implication from two given ones. Fuzzy Sets and Systems **205**, (2012), pp. 50–75. doi:10.1016/j.fss.2012.01.013

[3] Massanet, S., Torrens, J.: On the vertical threshold generation method of fuzzy implication and its properties, Fuzzy Sets and Systems **226** (2013), 32–52. doi:10.1016/j.fss.2013.03.003

[4] de Lima, A.A. et al.: Ordinal sums of the main classes of fuzzy negations and the natural negations of t-norms, t-conorms and fuzzy implications, International Journal of Approximate Reasoning, **116**, (2020) 19–32. doi:10.1016/j.ijar.2019.10.004

Generalized approach for construction of intuitionistic values

ĎUBEK Matúš
Department of Mathematics and Decriptive Geometry, Slovak University of Technology in Bratislava, Radlinského 11, 810 05 Bratislava, email: matus.dubek@stuba.sk

In this contribution, we describe a general framework for generating intuitionistic values by using automorphisms, i.e., generators of strong negations. A strong negation is a function $n\colon [0,1] \to [0,1]$ that is strictly decreasing, continuous and involutive. Every strong negation can be generated by an automorphism of the unit interval, i.e. a strictly increasing bijection, by the formula $n(x) = \varphi^{-1}(1 - \varphi(x))$ for all $x \in [0,1]$.

The class of all φ-intuitionistic values consists of pairs $(u, v) \in [0,1]^2$, which satisfies inequality $\varphi(u) + \varphi(v) \leq 1$. This condition is equivalent with $u \leq n(v)$, where n is the strong negation generated by φ. By using different automorphisms, different types of the intuitionistic values can be reconstructed, e.g., classical intuitionistic values proposed by Atanassov in [1] by choosing $\varphi(t) = t$, Pythagorean values proposed by Yager in [2] by choosing $\varphi(t) = t^2$, or, in general, q-rung values proposed in [3] by choosing $\varphi(t) = t^q$. A φ-intuitionistic fuzzy set A is then a mapping assigning a φ-intuitionistic value to every element of the universe of discourse, where u represents a membership and v represents a non-membership degree.

φ-intuitionistic sets are in a one-to-one correspondence with the class of all closed subintervals of the unit interval. It is convenient to use this relationship do define the usual operations over intuitionistic sets, by using generators of a t-(co)norm to define the operations over the intervals corresponding with given intuitionistic sets.

Acknowledgement.
The author of the paper acknowledges the support of the grant VEGA 1/0036/23 and VEGA 2/0128/24.

References

[1] Atanassov K. T. Intuitionistic Fuzzy Sets, VII ITKR Session, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: Int. J. Bioautomation, 2016, **20**(S1), S1-S6.

[2] Yager, R.R., Abbasov, A.M.: Pythagorean Membership Grades, Complex Numbers, and Decomposition Making. International Journal of Intelligent Systems **28**(5), (2013), pp. 436-452. ISBN: 987-456-12-12346. doi:10.1002/int.21584

[3] Yager, R.R.: Generalized orthopair fuzzy sets. IEEE Transactions on Fuzzy Systems **25**(5), (2017), pp. 1222-1230 p. 20. ISBN: 987-456-12-12346. doi:10.1109/TFUZZ.2016.2604005

Distributivity equation related to minitive and maxitive homogeneity of the upper n-Sugeno integral

HOVANA Anton · BOCZEK Michał · KAŁUSZKA Marek

Department of Applied Mathematics and Informatics, Technical university of Košice, 042-00 Košice, Slovakia, email: anton.hovana@tuke.sk · Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland, email: {michal.boczek.1, kaluszka}@p.lodz.pl

In the paper [2], sufficient conditions for minitive and maxitive homogeneity of the upper n-Sugeno integral have been stated. Both conditions can be expressed by

$$G(a \circ b, a \circ c) = a \circ G(b, c) \tag{2}$$

for all $a, b, c \in [0, \bar{y}]$, where $\circ \colon [0, \bar{y}]^2 \to [0, \bar{y}]$ is a uninorm on extended domain $[0, \bar{y}]$ defined by

$$a \circ b = \begin{cases} a \wedge b & \text{if } 0 \leq a, b \leq e, \\ a \vee b & \text{if } e \leq a, b \leq \bar{y}, \\ \psi(a, b) & \text{if } a \wedge b < e < a \vee b \end{cases} \tag{3}$$

with a fixed $e \in [0, \bar{y}]$ and fixed $\psi \in \{\wedge, \vee\}$, while $\bar{y} > 0$.

We will talk about a solution to equation (2). This problem belongs to the class of the distributivity equation which was firstly formulated by Aczél in 1966, see [1], and has not been completely solved for arbitrary functions G and $\circ$ so far. The contribution is devoted to a full characterization of functions G satisfying equation (2), while $\circ$ is defined by (3).

Acknowledgement.
The contribution was supported by the Slovak Research and Development Agency under contract No. APVV-21-0120 and VEGA 1/0243/23.

References

[1] Aczél J: *Lectures on Functional Equations and Their Applications.* Academic Press, 1966. ISBN: 978-0-12-043750-4.

[2] Boczek M., Hovana A., Hutník O., Kaluszka M.: New monotone measure-based integral inspired by scientific impact problem. European Journal of Operational Research **290**(1), (2021), pp. 346-357. https://doi.org/10.1016/j.ejor.2020.07.057.

Uncertainty modeling in music perception

HURAJOVÁ Kristína · HUTNÍK Ondrej
Institute of Mathematics, Pavol Jozef Šafárik University in Košice, Jesenná 5, 040 01 Košice, Slovakia,
email: kristina.hurajova@student.upjs.sk · ondrej.hutnik@upjs.sk

"Uncertainty in music perception" refers to the cognitive processes involved in how individuals perceive and interpret musical stimuli in the presence of various forms of uncertainty. Since uncertainty can arise from different sources and can affect different aspects of music perception, in our contribution we focus only on two key aspects: pitch perception, and harmonic perception. Uncertainty in pitch perception refers to difficulties in accurately perceiving or identifying the pitch of musical tones, especially when tones are ambiguous or when there is noise in the auditory environment. On the other hand, uncertainty in harmonic perception involves the interpretation of harmonic relationships, chord progressions, and tonal structures in music. This includes understanding how listeners perceive tonal stability, chord changes, and harmonic tension-resolution patterns. There are several approaches to modeling uncertainty in music including probabilistic models, stochastic processes, interval-based representations, machine learning methods, etc. Here we aim to develop the fuzzy approach.

Let $\mathbb{T}$ be a set of tones, cf. [2]. The function $\varphi : \mathbb{T} \to \Lambda$, where Λ is a commutative l-semigroup, is called a *pitch function.* Hence, $\varphi(\tau) \in \Lambda$ is a pitch of the tone $\tau \in \mathbb{T}$. Often $\Lambda = \mathbb{R}$ (the set of all real numbers) or $\mathbb{Z}$ (the set of all integer numbers) or $\mathbb{Q}$ (the set of all rational numbers) or $\mathbb{Q}_{p,q}$ ("spirals of the fifths", i.e., the set of all numbers of the type $p^\alpha q^\beta$, where α, β are rational numbers and p, q are two algebraic numbers). We consider $\Lambda = \mathbb{F}_\triangle$, the set of all triangular fuzzy numbers, which originates from the work of Garbuzov [1]. Garbuzov revolutionized the study of musical intervals by proposing the concept of musical "zones". For each note in a score, there exists a set of possible tones with frequencies that form the Garbuzov zone of this note. This describes a method for introducing a fuzzy set structure into the consideration of tone systems. The triangular symmetrical fuzzy number $\varphi(\tau)$ (we identify the fuzzy number with its membership function)

$$\varphi(\tau)(x) = \Phi(\xi(\tau), \delta(\tau))(x) = \begin{cases} 1 - \frac{|\xi(\tau)-x|}{\delta(\tau)}, & \text{if } |\xi(\tau) - x| < \delta(\tau), \\ 0, & \text{otherwise,} \end{cases}$$

is called the *fuzzy tone pitch*, see Fig. 1. This imagination enables to define several measures of similarity between tones and compatibility of tone systems, see [3, 5].

Using the fuzzy tone pitch, the uncertainty in harmonic perception of music can be modeled by (fuzzy) distances between chords. In music, a chord is a group of three or more tones sounded together simultaneously. Although computing a distance between two chords of the same cardinality is quite easy, there are several approaches to evaluating distances between chords of different cardinality, as summarized in [4], in which some properties of the metric may fail. We demonstrate how the fuzzy approach can be useful in addressing the question of distinguishing among musical chords that are indistinguishable in the crisp-set context.

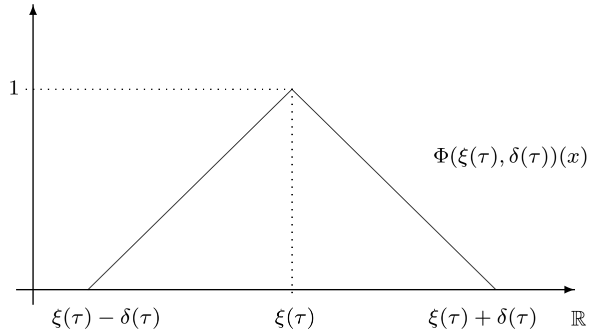


Figure 1: Fuzzy pitch $\varphi(\tau)(x) = \Phi(\xi(\tau), \delta(\tau))(x)$ of a tone $\tau \in \mathbb{T}$

Acknowledgement.
This work was supported by the Slovak Research and Development Agency under the contract No. APVV-21-0468, and the grant VEGA 1/0657/22.

References

[1] Garbuzov, N.A.: *The Zonal Nature of the Human Aural Perception* (in Russian). Izdatelstvo Akademii Nauk SSSR, Moscow-Leningrad, 1948.

[2] Haluška, J.: *Mathematical Theory of Tone Systems*. Marcel Dekker, New York, 2004.

[3] Haluška, J., Hutník, O.: On algebras of symmetrical associative aggregation operators related to means. Tatra Mt. Math. Publ. **34** (2006), pp. 119–133.

[4] Hurajová, K., Hutník, O.: Towards measuring the distances of chords of different cardinalities. In: *Mathematics and Computation in Music 2024* (to appear).

[5] Liern, V.: Fuzzy tuning systems: the mathematics of musicians. Fuzzy Sets and Systems **150** (2005), pp. 35–52.

Residual implications on simple lattices

JANIŠ Vladimír
Faculty of Science, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia, email: vladimir.janis@umb.sk

A fuzzy implication as a counterpart of an implication in multivalued logic is a mapping $I : [0,1]^2 \to [0,1]$ with properties $I(0,0) = I(0,1) = I(1,1) = 1, I(1,0) = 0$, decreasing in the first and increasing in the second variable. A comprehensive study of such functions can be found in [1] and a number of papers. However, mostly the implications on the unit interval or bounded chains are discussed.

Our research is aimed at the study of fuzzy implications on some simple cases of finite bounded lattices, which are not chains, such as horizontal sums of chains or lattices similar to horizontal sums, with a single atom or a single coatom.

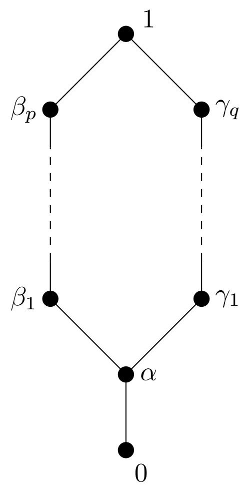


Figure 2: A lattice with an atom α

An example of a lattice with two pasted chains and an atom is in the Figure 1, in a similar way we can construct a lattice with a coatom or both atom and a coatom.

As the set of all implications on such lattices is extremely rich, we restrict ourselves on residual ones based on a triangular norm T, which are the mappings

$$I_T(x,y) = \sup\{z \in L; T(x,z) \leq y\}.$$

For these lattices we present the description of residual implications as a table that enables us to find the values of a residual implication for a given pair of elements, we estimate their number depending on a given triangular norm or at least we set the upper bound for their cardinality. Similar estimations have been done in [2], however only for finite chains.

Acknowledgement.
The research was supported by grant 1/0124/24 provided by the Slovak grant agency VEGA.

References

[1] Baczyński, M., Jayaram, B.: *Fuzzy Implications*. Springer, 2008. ISBN: 978-3-540-69080-1.

[2] Mas, M., Monserrat, M., Torrens, J.: S-implications and R-implications on a finite chain. Kybernetika **40**, (2004), pp. 3-20. ISSN 1805-949X.

A Choquet-like operator for aggregating multivalued data

JÓZEFIAK Tomasz · BOCZEK Michał · KAŁUSZKA Marek · OKOLEWSKI Andrzej
Institute of Mathematics, Lodz University of Technology, 93-590 Lodz, Poland, email:
tomasz.jozefiak@dokt.p.lodz.pl · {michal.boczek.1, marek.kaluszka, andrzej.okolewski}@p.lodz.pl

We propose a novel generalization of the classical discrete Choquet integral to the multivalued framework in terms of an admissible order that refines the natural partial order on the considered value set. The newly introduced Choquet-like operator takes as input a finite number of values of a given type, in particular real numbers, intervals, and vectors, and returns a single output value of the same type as the input values. Our operator encompasses many extensions of the discrete Choquet integral known in the literature, both in the real number setting [1, 3] and in the interval setting [2, 4]. We give necessary and sufficient conditions for the operator to be well-defined, that is, invariant with respect to permutations of the input data, and to be monotone with respect to the admissible order. We then provide a complete characterization of the Choquet-like operator as an aggregation function with respect to the admissible order and study its selected special cases.

References

[1] Boczek, M., Józefiak, T., Kaluszka, M., Okolewski, A.: On the monotonicity of the discrete Choquet-like operators. International Journal of Approximate Reasoning **163**, (2023), 109045. ISSN 0888-613X. doi:https://doi.org/10.1016/j.ijar.2023.109045

[2] Bustince, H., Galar, M., Bedregal, B., Kolesárová, A., Mesiar, R.: A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy set applications. IEEE Transactions on Fuzzy Systems **21**(6), (2013), pp. 1150-1162. ISSN 1063-6706. doi:10.1109/TFUZZ.2013.2265090

[3] Dimuro, G.P., Fernández, J., Bedregal, B., Mesiar, R., Sanz, J.A., Lucca, G., Bustince, H.: The state-of-art of the generalizations of the Choquet integral: From aggregation and preaggregation to ordered directionally monotone functions. Information Fusion **57**, (2020), pp. 27-43. ISSN 1566-2535. doi:https://doi.org/10.1016/j.inffus.2019.10.005

[4] Takáč, Z., Uriz, M., Galar, M., Paternain, D., Bustince, H.: Discrete IV d_G-Choquet integrals with respect to admissible orders. Fuzzy Sets and Systems **441**, (2022), pp. 169-195. ISSN 0165-0114. doi:https://doi.org/10.1016/j.fss.2021.09.013

Pseudo-uninorms on bounded lattices where all points are comparable with neutral element

KALAFUT Juraj · MESIAROVÁ-ZEMÁNKOVÁ Andrea

Department of Mathematics and Decriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovakia, email: juraj.kalafut@stuba.sk · Mathematical Institute, Slovak Academy of Science, Štefánikova 49, 814 73 Bratislava, email: zemankova@mat.savba.sk · Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, CE IT4Innovations, 30.dubna 22, 701 03 Ostrava, Czech Republic

In this contribution we would like to discuss an extension of pseudo-uninorms, which are non-commutative counterparts of uninorms, to a bounded lattice. This work is a continuation of the research on pseudo-uninorms on the unit interval [1, 2].

The initial characterization of idempotent uninorms on specific complete lattices was provided in [3]. This characterization was later extended to general bounded lattices via uninorm-induced order in [4, 5]. Based on these results, we show that each idempotent uninorm on a bounded lattice with a neutral element e that is comparable with all elements of the lattice can be constructed as a Clifford's ordinal sum of join and meet semilattices, which are determined by the corresponding bounded lattice. Therefore two different idempotent uninorms on such a lattice differ only in the linear order used in the corresponding ordinal sum of these semilattices.

Unlike idempotent pseudo-uninorms on the unit interval, idempotent pseudo-uninorms on a bounded lattice may not be internal. Additionally, an idempotent pseudo-uninorm on $[0, e]^2$ coincides with the meet, while on $[e, 1]^2$ it coincides with the join. We characterize idempotent pseudo-uninorms with a neutral element e that is comparable with all elements of the corresponding bounded lattice. On the one hand we show that the points of non-commutativity belong to the set $A(e)$. On the other hand we show that each idempotent pseudo-uninorm on a such bounded lattice can be decomposed via Clifford's ordinal sum into semilattices and projections to one of the coordinates on the two-element set.

Acknowledgement.
This contribution was supported by grants VEGA 1/0036/23, VEGA 2/0128/24 and Program na podporu mladých výskumníkov (Young Researchers Support Programme).

References

[1] Mesiarová-Zemánková, A.: Decomposition of idempotent pseudo-uninorms via ordinal sum. Information Sciences **648**, (2023). ISSN 1872-6291. https://doi.org/10.1016/j.ins.2023.119519

[2] Mesiarová-Zemánková, A., Kalafut, J.: Pseudo-uninorms with continuous Archimedean underlying functions. Fuzzy Sets and Systems **471**, (2023). ISSN 0165-0114. https://doi.org/10.1016/j.fss.2023.108674

[3] Drygaś, P.: Some Remarks About Idempotent Uninorms on Complete Lattice. In *Kacprzyk, J., Szmidt, E., Zadrożny, S., Atanassov, K., Krawczak, M. (eds) Advances*

in Fuzzy Logic and Technology 2017. EUSFLAT IWIFSGN 2017 2017. Advances in Intelligent Systems and Computing, vol 641. Springer, Cham. ISBN 978-3-319-66830-7. https://doi.org/10.1007/978-3-319-66830-7_57

[4] Su, Y., Wang, Z., Mesiarová-Zemánková, A., Mesiar, R.: Characterizing three classes of idempotent uninorms on a bounded lattice. Iranian Journal of Fuzzy Systems **20** (5), pp. 109–120 (2023). ISSN 2676-4334. https://doi.org/10.22111/IJFS.2023.7685

[5] Mesiarová-Zemánková, A., Mesiar, R., Su, Y., Wang, Z.: Idempotent uninorms on bounded lattices with at most a single point incomparable with the neutral element (under review).

Ordinal sum of abelian semigroups

KALINA Martin
Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Radlinského 11, email: martin.kalina@stuba.sk

This is a continuation of the contribution presented at FSTA 2024 in Liptovský Ján [4]. At FSTA, the author presented an ordinal sum construction (see [1]) of t-subnorms. Any t-subnorm has at least 1 idempotent element, namely 0. Motivated by a question posted by Paweł Drygaś, in this contribution a new generalization is presented. An ordinal sum construction of commutative semigroups will be presented. The author comes back to the Clifford's ordinal sum construction [1] utilizing also the z-ordinal sum provided by Mesiarová-Zemánková [5], a generalization of the Clifford's ordinal sum construction showing by Hliněná and Kalina [3] and also some ideas by De Baets and Mesiar [2].

The aim in this contribution is to present new construction possibilities yielding uninorms. It goes on bringing some new bricks in constructions for uninorms.

Acknowledgement.
The author was supported from the VEGA grant agency, grant Nrs. 1/0036/23 and 2/0128/24.

References

[1] Clifford, A.H.: Naturally totally ordered commutative semigroups. Amer. J. Math. **76** (1954), pp. 631-–646.

[2] De Baets, B., Mesiar, R.: Ordinal sums of aggregation operators. In *Bouchon-Meunier, B., Gutiérrez-Ríos, J., Magdalena, L., Yager R.R. (Eds.), Technologies for Constructing Intelligent Systems, vol. 2: Tools.* Physica-Verlag, Heidelberg (2002), pp. 137–148.

[3] Hliněná, D., Kalina, M.: A new construction for t-norms and their application to an open problem of Alsina, Frank and Schweizer. Fuzzy Sets and Systems. **451** (2022), pp. 16–27. ISSN: 0165-0114 `doi:https://doi.org/10.1016/j.fss.2022.06.002`

[4] Kalina, M.: Construction for uninorms on the unit interval from t-subnorms and/or t-superconorms. In *FSTA 2024: Liptovský Ján, Slovakia, January 28 - February 2, 2024: Book of Abstracts of the Seventeenth International Conference on Fuzzy Set Theory and Applications*, Ostrava : University of Ostrava, (2024), pp. 53–54. ISBN 978-80-7599-416-5.

[5] Mesiarová-Zemánková, A.: Characterization of uninorms with continuous uderlying t-norm and t-conorm by means of the ordinal sum construction. International Journal of Approximate Reasoning. **83** (2017), pp. 176–192. ISSN: 1873-4731 `doi:http://dx.doi.org/10.1016/j.ijar.2017.01.007`

Rational divergence measures

KOBZA Vladimír
Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, email: vladimir.kobza@umb.sk

In our work we have extended the concept of measuring differences between two fuzzy subsets defined on a finite universe. An alternative approach to the divergence measures is motivated by class of the rational similarity measures between fuzzy subsets expressed using some set operations (namely intersection, complement, difference and symmetric difference) and their scalar cardinalities. Some open problems are discussed in this contribution.

A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a divergence measure if and only if the function D satisfies the following conditions:

(1) for all $A \in \mathcal{F}(X)$; $D(A, A) = 0$;

(2) for all $A, B \in \mathcal{F}(X)$; $D(A, B) = D(B, A)$;

(3) for all $A, B, C \in \mathcal{F}(X)$; $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$.

We recall that the binary fuzzy relation R defined on $X \times X$ is a T-equivalence if and only if it satisfies the following properties for each $x, y, z \in X$ and the t-norm T:

(a) reflexivity: $R(x, x) = 1$,

(b) symmetry: $R(x, y) = R(y, x)$,

(c) T-transitivity: $T(R(x, y), R(y, z)) \leq R(x, z)$.

We have studied some class of rational similarity measures which are also T-equivalences [1]. All of them have been modified and we have introduced the system of rational measures derived from T-equivalences [3]. In the second step, all measures were fuzzified for the triple (X, T, S), where $|X| = n$, $T = T_M$ and $S = S_M$. Moreover, for the values p from Frank's family of t-norms T_p^F some restriction conditions must be considered [2]. In our future work, we will focus on the other parametric systems of t-norms. We also have studied some important properties of these divergences focused on the monotonicity, boundary conditions and local property.

References

[1] De Baets, B., De Meyer, H., Naessens, H.: A class of rational cardinality-based similarity measures. J. Comput. Appl. Math. **132**, (2001), pp. 51-69. DOI:10.1016/s0377-0427(00)00596-3

[2] Kobza, V.: Divergence measure between fuzzy sets using cardinality. Kybernetika **53**(3), (2017), pp. 418-436. ISSN print 0023-5954, ISSN online 1805-949X. 10.14736/kyb-2017-3-0418.

[3] Wygralak, M.: *Cardinalities of Fuzzy Sets*. Springer, Berlin, Heidelberg, New York, 2003. ISBN: 978-3-540-36382-8. DOI:10.1007/978-3-540-36382-8

Idea privacy of data in early detection of the risk of depressive episodes concerning uncertainty measures

PĘKALA Barbara · KOSIOR Dawid · CZUMA Janusz · SZKOŁA Jarosław
University of Rzeszów, Rzeszów, Poland, email: {bpekala, dkosior, jszkola}@ur.edu.pl · LUX MED - PROFEMED Medical Center, Rzeszów, Poland, email: ddcad55@gmail.com

The core of the proposed methodology is based on the description of possible and necessary measures in epistemic issues, especially optimistic (possible) and pessimistic (necessary) entropy measures. Our main motivation is to use our concept of measures to early detect the risk of depressive episodes concerning privacy data [1, 2, 3] or different kinds of data in sociological or finance or other issues. We propose an effective algorithm to estimate the risk of depressive episodes. By using the concept of federation learning and a more sensitive algorithm to diagnosis we aim to identify the risk of depressive states before they are indicated in other tests, which can prevent the appearance of symptoms. The proposed methodology, in the form of a system, may use data from different medical centers without sharing data.

References

[1] Kairouz, P., McMahan, B.: Advances and open problems in federated learning. Foundations and Trends@ in Machine Learning, **14**, (2021) pp. 1–210.

[2] Konečný, J., McMahan, H. B., Ramage, D., and Richtárik, P.: Federated optimization: Distributed machine learning for on-device intelligence. ArXiv, (2016) 1610.02527.

[3] Konečný, J., McMahan, H. B., Yu, F. X., Richtárik, P., Suresh, A. T., and Bacon, D.: Federated learning: Strategies for improving communication efficiency. ArXiv, (2017) 1610.05492.

[4] Pękala, B., Garwol, K., Czuma, J., Kosior, D., Zaręba, L., Chyła, M.,: Early detection of the risk of depressive episodes using a proprietary diagnostic test by new epistemic similarity measures, Applied Soft Computing **148**, (2023) 110910.

On compact elements in lattices of aggregation functions

PÓCS Jozef
Mathematical Institute Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia; email: pocs@saske.sk

Several important classes of aggregation functions [1] defined on a bounded lattice form a lattice with respect to the pointwise operations of join and meet, respectively. An element a in a lattice L is compact if for every subset $A \subseteq L$, $a \leq \bigvee A$ implies $a \leq A'$ for some finite $A' \subseteq A$. The lattice L is compactly generated if every element is a join of compact elements, and a lattice is algebraic if it is complete and compactly generated. Particularly, if L is an algebraic lattice and L_C is the set of all compact elements of L, then L_C forms a join semi-lattice with 0 and $L \cong \mathrm{Id}(L_C)$, i.e., the lattice L is isomorphic to the lattice of all ideals of the semi-lattice L_C.

Algebraic lattices are very important in the study of various algebraic structures. These types of lattices have been studied in detail and have several interesting properties.

The aim of this paper is to describe compact elements in the lattices of some classes of aggregation functions which are defined within the framework of an algebraic lattice L. We also show that aggregation functions which are compact can be seen as a suitable approximation of other aggregation functions.

Acknowledgement.
This work was supported by the Scientific Grant Agency (VEGA) under grant no. VEGA 2/0104/24.

References

[1] Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: *Aggregation Functions.* Cambridge University Press, Cambridge, 2009. ISBN: 978-0-521-51926-7. doi.org/10.1017/CBO9781139644150

On polynomial 3-copulas

ŠELIGA Adam · SAMINGER-PLATZ Susanne · KOLESÁROVÁ Anna · MESIAR Radko · KLEMENT Erich Peter

Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Bratislava, Slovakia, email: {adam.seliga,anna.kolesarova,radko.mesiar}@stuba.sk · Institute for Mathematical Methods in Medicine and Data Based Modeling, Johannes Kepler University, Linz, Austria, email: {susanne.saminger-platz,ep.klement}@jku.at

Copulas capture the essence of dependency structures by separating the marginal distributions of the random variables involved in the joint distribution function. The research of scientists is currently notably focused on the study of two-dimensional copulas. Higher dimensional copulas, i.e., n-copulas where $n \geq 3$, extend the concept of the bivariate copula functions to model relationships among three or more variables allowing them to be modelled in a more nuanced and comprehensive manner. Despite their potential, their application has been restrained by the mathematical complexities and computational challenges they entail. This has particularly impacted the broader adoption and exploration of these models in various fields of research.

In this contribution, the study of the polynomial 3-copulas is started extending the foundational work on bivariate copulas demonstrated in the previous studies, see, e.g., [1]. We introduce a polynomial-based approach for constructing three-dimensional copulas and examine their dependence parameters such as the Spearman's rho, Kendall's tau, Blomqvist's beta or Gini's gamma.

In particular, we recall a complete characterization of 3-copulas which are polynomials of degree 5 and discuss the case of polynomials of degree 6 (observe that here a complete description of related 3-copulas is still an open problem).

Acknowledgement.
Adam Šeliga was supported by the Slovak University of Technology project SEMPIa entitled Identifikácia materiálov v zbierkach kultúrneho dedičstva na báze syntetických plastov prostredníctvom ich charakteristických ,,odtlačkov prstov". The support of the grant VEGA 1/0036/23 is also acknowledged.

References

[1] Šeliga, A., Kauers, M., Saminger-Platz, S., Mesiar, R., Kolesárová, A., Klement, E.P.: Polynomial bivariate copulas of degree five: characterization and some particular inequalities. Dependence Modeling **9**(1) (2021), 13-42. ISSN 2300-2298. doi:10.1515/demo-2021-0101

Aggregating on discrete chains based on extreme values

STUPŇANOVÁ Andrea
STU in Bratislava, Faculty of Civil Engineering, Radlinského 11, 810 05 Bratislava, Slovakia, email: andrea.stupnanova@stuba.sk

Let C be a finite discrete chain of numerical values or linguistic terms, represented by the set $\{a_0, a_1, \ldots, a_d\}$, with a linear ordering $\prec$ such that $a_0 \prec a_1 \prec a_2 \prec \cdots \prec a_d$. The focus of this study is on extended aggregation functions on a chain C such that the output value depends only on a small set of largest and/or smallest values of the given inputs.

Definition 1 *Let (u, v) be a couple of non-negative integers. An extended discrete aggregation function $A : \bigcup_{n=1}^{\infty} C^n \to C$ is called a (u,v)-aggregation function whenever for any $n > u + v$ and $\mathbf{x}$ from C^n it holds*

$$A(\mathbf{x}) = A(x_{(1)}, ..., x_{(u)}, x_{(n-v+1)}, ..., x_{(n)}), \tag{4}$$

where $(.)$ is a permutation such that $x_{(1)} \leq \cdots \leq x_{(n)}$, i.e., for evaluation of $A(\mathbf{x})$ only u smallest and v greatest values of the n-tuple $\mathbf{x}$ are considered.

After presenting some general construction methods, we proceed to study in detail the most distinguished binary associative functions that possess a neutral element e and their link to (u, v)-aggregation functions. The smallest t-conorm $S_M = \max$ is (in its extended form) just a (0,1)-aggregation function, and it is the only discrete t-conorm with this property. The drastic sum S_W is (0,2)-aggregation function. We show that any extended t-conorm S is a $(0, v)$-aggregation functions, where $v \in \{1, \ldots, d\}$. Furthermore, it is shown that the unique t-conorm which is $(0, d)$-aggregation function is the Lukasiewicz t-conorm S_L. As an important consequence, we can introduce a classification of discrete t-conorms with exactly d-classes based on the related parameter v. We present some construction methods for $(0, v)$-t-conorms. By duality, results for discrete t-conorms could be rewritten for discrete t-norms. Results obtained for discrete t-conorms and discrete t-norms on C, considering the influence of outliers, could be applied to the discrete uninorms and nullnorms.

Acknowledgement.
The support of the grant VEGA 1/0036/23 is kindly announced.

References

[1] Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: *Aggregation Functions*. Cambridge University Press, Cambridge, 2009. ISBN: 978-0-521-51926-7. doi.org/10.1017/CBO9781139644150

[2] Klement, E.P., Mesiar, R.. Pap, E.: *Triangular norms*. Kluwer Academic Publishers, Dordrecht, 2000. ISBN:0-7923-6416-3.

[3] Mesiar, R., Kolesárová, A., Stupňanová, A.: On a new classification of triangular norms. Fuzzy Sets and Systems **466** (2023), art. no. 108393. ISSN 0165-0114. `10.1016/j.fss.2022.09.002`

Uncertainty Modeling 2024

Book of Abstracts

Edited by:	prof. RNDr. Ondrej Hutník, PhD.
Publisher:	Pavol Jozef Šafárik University in Košice Publishing ŠafárikPress
Year:	2024
Pages:	31
Author's sheets:	1,2
Edition:	first

ISBN 978-80-574-0318-0 (e-publication)